

# GLOBAL MONGE-AMPÈRE EQUATION WITH ASYMPTOTICALLY PERIODIC DATA

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**ABSTRACT.** Let  $u$  be a convex solution to  $\det(D^2u) = f$  in  $\mathbb{R}^n$  where  $f \in C^{1,\alpha}(\mathbb{R}^n)$  is asymptotically close to a periodic function  $f_p$ . We prove that the difference between  $u$  and a parabola is asymptotically close to a periodic function at infinity, for dimension  $n \geq 3$ .

## 1. INTRODUCTION

In this article we study convex, entire viscosity solutions  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ , to the Monge-Ampère equation

$$(1.1) \quad \det(D^2u) = f(x), \quad \text{in } \mathbb{R}^n.$$

The forcing term  $f$  is assumed to be positive and asymptotically close to a periodic function at infinity. Our main goal is to establish a classification theorem for such solutions.

Monge-Ampère equation with periodic data can be found in various topics in applied mathematics such as homogenization theory, optimal transportation problems, vorticity arrays, etc. Equation (1.1) also appears in differential geometry, when it is lifted from a Hessian manifold [12]. In spite of the profusion in application, Monge-Ampère equation is well known for its analytical difficulty and it is no exception for equation (1.1) when the right hand side is close to a periodic function. In [9] Caffarelli and Li proved that if  $f$  is a positive periodic function,  $u$  has to be a parabola plus a periodic function with the same periodicity of  $f$ . This theorem can be viewed as an extension to the classification theorems of Jörgens [28], Calabi [13], Pogorelov [35], Caffarelli-Li [8] for the Monge-Ampère equation  $\det(D^2u) = 1$ .

The aim of this article is to establish an optimal perturbation result from the Caffarelli-Li's classification theorem, as to cover forcing terms  $f$  that are asymptotically a periodic function at infinity.

In more precise terms, we let

$$(1.2) \quad \mathcal{A} \quad \text{be the set of all positive definite and symmetric matrices,}$$

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the assumption on  $f$  is as follows: Let  $f_p$  be a positive,  $C^{1,\alpha}$  periodic function in  $\mathbb{R}^n$ , i.e.:

$$(1.3) \quad \begin{aligned} &\exists d_0 > 0, \alpha \in (0, 1), a_1, \dots, a_n > 0 \text{ such that} \\ &d_0^{-1} \leq f_p \leq d_0, \quad \|f_p\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq d_0, \\ &f_p(x + a_i e_i) = f_p(x), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

where  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ . We assume that  $f \in C^{1,\alpha}(\mathbb{R}^n)$  is asymptotically close to  $f_p$  in the following sense:

$$(1.4) \quad \begin{aligned} &\exists d_1 > 0 \text{ and } \beta > 2, \text{ such that} \\ &d_1^{-1} \leq f(x) \leq d_1, \quad \forall x \in \mathbb{R}^n, \quad \|f\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq d_1, \\ &|\nabla^j(f^{\frac{1}{n}} - f_p^{\frac{1}{n}})(x)| \leq d_1(1 + |x|)^{-\beta-j}, \quad \forall x \in \mathbb{R}^n, \quad j = 0, 1, 2, 3. \end{aligned}$$

**Remark 1.1.**  $f, f_p \in C^{1,\alpha}(\mathbb{R}^n)$  but the difference between  $f^{\frac{1}{n}}$  and  $f_p^{\frac{1}{n}}$  is more smooth.

Under the above framework, our main theorem is:

**Theorem 1.1.** *Let  $n \geq 3$  and  $u \in C^{3,\alpha}(\mathbb{R}^n)$  be a convex solution to*

$$(1.5) \quad \det(D^2 u) = f, \quad \text{in } \mathbb{R}^n$$

*where  $f$  satisfies (1.4). Then there exist  $b \in \mathbb{R}^n$ ,  $A \in \mathcal{A}$  (defined in (1.2)) with  $\det(A) = \bar{f}_{\Pi_{1 \leq i \leq n}[0, a_i]} f$ , and  $v \in C^{3,\alpha}(\mathbb{R}^n)$ , which is  $a_i$ -periodic in the  $i$ -th variable, such that*

$$(1.6) \quad \left| u(x) - \left( \frac{1}{2} x' A x + b \cdot x + v(x) \right) \right| \leq C(1 + |x|)^{-\sigma}, \quad \forall x \in \mathbb{R}^n$$

*for some  $C(d_0, d_1, n, \beta, a_1, \dots, a_n) > 0$  and  $\sigma := \min\{\beta, n - 2\}$ .*

Here we note that  $\bar{f}_{\Pi_{1 \leq i \leq n}[0, a_i]} f$  is the average of  $f$  over the box.

**Remark 1.2.** *Theorem 1.1 does not imply corresponding (better) estimates on higher order derivatives because of the oscillation of  $D^2 u$ .*

Caffarelli and Li [9] proved that

$$u(x) = \frac{1}{2} x' A x + b \cdot x + v(x), \quad \mathbb{R}^n, \quad n \geq 2$$

if  $f = f_p$  in (1.1). Thus Theorem 1.1 is an extension of the theorem of Caffarelli-Li.

The assumption  $\beta > 2$  is essentially optimal, as one can observe from the following example: let  $f$  be a radial, smooth, positive function such that  $f(r) \equiv 1$  for  $r \in [0, 1]$  and  $f(r) = 1 + r^{-2}$  for  $r > 2$ . Let

$$u(r) = n^{\frac{1}{n}} \int_0^r \left( \int_0^s t^{n-1} f(t) dt \right)^{\frac{1}{n}} ds, \quad r = |x|.$$

It is easy to check that  $\det(D^2u) = f$  in  $\mathbb{R}^n$ . Moreover for  $n \geq 3$ ,

$$u(x) = \frac{1}{2}|x|^2 + O(\log|x|)$$

at infinity, which means by taking  $f_p \equiv 1$  the estimate in Theorem 1.1 is violated for  $n \geq 4$ .

One major difficulty in the study of (1.1) is that the right hand side oscillates wildly when it is scaled. The regularity theory for Monge-Ampère equations with oscillating right hand side is very challenging (see [18, 19]). In this respect, we found that the strategy implemented by Caffarelli and Li in [9] is nearly perfect, as we can only simplify a small part of their argument in [9]. In turn, our arguments are essentially based on the corresponding steps from [9] as well as previous works of Caffarelli and Li [5, 6, 8]. The main difference, though, is that in the proof of Theorem 1.1 one needs to take care of perturbational terms in a sharp manner. In order to handle all the perturbations, we need to make use of intrinsic structures implied by Monge-Ampère equations (such as (2.10) below), estimates on Green's functions by Littman-Stampacchia-Weinberger [33] and Krylov-Safonov Harnack inequalities, etc.

Theorem 1.1 is closely related to the exterior Dirichlet problem: Given a strictly convex set  $D$  and the value of  $u$  on  $\partial D$ , can one solve the Monge-Ampère equation in  $\mathbb{R}^n \setminus D$  if the asymptotic behavior of  $u$  at infinity is prescribed? Clearly Theorem 1.1 must be established before such a question can be attacked. We plan to address the exterior Dirichlet problem in a future work. The traditional (interior) Dirichlet problem has been fairly well understood through the contribution of many people (see [1, 2, 34, 13, 35, 14, 11, 6, 24, 25, 26, 30, 27, 37, 39, 19, 18] and the references therein). If  $f$  is equal to a positive constant outside a compact set, Delanoë [17], Ferrer-Martínez-Milán [20, 21] and Bao-Li [3] studied the exterior Dirichlet problem for  $n = 2$ , Caffarelli-Li [8]) studied the case of  $n \geq 3$ . If  $f$  is a perturbation of a positive constant at infinity, Bao-Li-Zhang [3] studied the exterior Dirichlet problem in [4].

The organization of this article is as follows: The proof of Theorem 1.1 consists of five steps. First in step one we employ the argument in [6, 8] to show that the growth of the solution of (1.1) is roughly similar to that of a parabola. Then in step two we prove that  $D^2u$  is positive definite, which makes (1.1) uniformly elliptic. The key point in this step is to consider a second order incremental of  $u$  as a subsolution to an elliptic equation. In step three we prove a pointwise estimate of the second incremental of  $u$ . The proof of Theorem 1.1 for  $n \geq 4$  is placed in step four since all the perturbations in this case are bounded. Finally in step five we prove the case  $n = 3$ , which is a little different because of a logarithmic term. We use the Krylov-Safonov Harnack inequality for linear equations to overcome the difficulties caused by the logarithmic term.

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## 2. PROOF OF THEOREM 1.1

Since the Monge-Ampère equation is invariant under affine transformation, we assume  $a_1 = \dots = a_n = 1$  and  $\int_{[0,1]^n} f = 1$ .

In step one we prove that  $u$  grows like a quadratic polynomial at infinity. First we normalize  $u$  to make  $u(0) = 0$  and  $u \geq 0$  in  $\mathbb{R}^n$ . Since  $f$  is bounded above and below by two positive constants, we use the argument in Caffarelli-Li [8], see also [4]. Let

$$\Omega_M = \{x \in \mathbb{R}^n, \quad u(x) < M\}.$$

Then the following properties hold:

- (1)  $C^{-1}M^{\frac{n}{2}} \leq |\Omega_M| \leq CM^{\frac{n}{2}}$ ,
- (2) There exists  $A_M(x) = a_M \cdot x + b$  such that  $\det(a_M) = 1$  and  $B_R \subset A_M(\Omega_M) \subset B_{nR}$  and
- (3)  $\frac{1}{C}\sqrt{M} \leq R \leq C\sqrt{M}$ ,
- (4)  $2nR \geq \text{dist}(a_M(\Omega_{\frac{M}{2}}), \partial a_M(\Omega_M)) \geq \frac{R}{C}$

where all the constants  $C$  only depend on  $d_1$  and  $n$ .

All the properties listed above are proved in [8] only based on the assumption that  $f$  is bounded above and below by two positive constants. Here for the convenience of the reader we mention the idea of the proof: First Caffarelli-Li used the following lemma (Lemma 2.9 in [8])

*Lemma A (Caffarelli-Li):* Let  $e_1 = (1, 0, \dots, 0)$  and

$$B'_\delta = \{(0, x_2, \dots, x_n); \quad |(0, x_2, \dots, x_n)| < \delta\}.$$

Let  $K$  be the convex hull of  $\bar{B}'_\delta \cup \{re_1\}$ , and let  $u$  be a nonnegative convex viscosity solutions of  $\det(D^2u) \geq \lambda > 0$  in the interior of  $K$ . Assume  $u \leq \beta$  on  $\bar{B}'_\delta$ . Then there exists  $C(n) > 1$  such that

$$\max\{\beta, u(re_1)\} \geq \frac{\lambda^{1/n} \delta^{\frac{2(n-1)}{n}} r^{2/n}}{C}.$$

In other words,  $u$  cannot be small in one direction for too long. Lemma A is important since it says Pogorelov's famous example of non-strictly convex solution does not exist if the domain is large. A simple application of Lemma A leads to  $\Omega_M \subset B_{CM^{\frac{n}{2}}}$ . A volume preserving affine transformation can be used to make the image of  $\partial\Omega_M$  between two balls with comparable radii. A comparison with a parabola gives  $|\Omega_M| \sim M^{\frac{n}{2}}$  using only the upper bound and lower bound of  $f$ . In [5] Caffarelli proved that  $u$  must depart from its level set in a non-tangential manner, using this we have

$dist(A_M(\partial\Omega_M), A_M(\Omega_{M/2})) \sim M^{\frac{1}{2}}$  where  $A_M$  is a volume-preserving affine transformation:

$$A_M(x) = a_M x + b_M, \quad \det(a_M) = 1.$$

Using  $u(0) = 0$  and  $u \geq 0$  one can further conclude

$$(2.1) \quad B_{R/C} \subset a_M(\Omega_M) \subset B_{2nR} \quad \text{where} \quad R = M^{\frac{1}{2}}.$$

Equation (2.1) makes it convenient to define

$$u_M(x) = \frac{1}{R^2} u(a_M^{-1}(Rx)), \quad x \in O_M := \frac{1}{R} a_M(\Omega_M),$$

Using  $u(0) = 0$  and  $u \geq 0$  we have

$$B_{1/C} \subset O_M \subset B_{2n}.$$

In order to analyze the level surfaces of  $u_M$ , which satisfies

$$(2.2) \quad \det(D^2 u_M(\cdot)) = f(a_M^{-1}(R\cdot)), \quad \text{in } O_M.$$

We use the following equation to give a good approximation of  $u_M$ :

$$(2.3) \quad \begin{cases} \det(D^2 w_p) = f_p(a_M^{-1}(Rx)), & \text{in } O_M, \\ w_p = M/R^2, & \text{on } \partial O_M. \end{cases}$$

Let  $h = u_M - w_p$  be the difference of  $u_M$  and  $w_p$ , then by the Alexandrov estimate ( see [8, 4]) we have

$$\max_{O_M} (h^-) \leq C \left( \int_{S^+} \det(D^2(u_M - w_p)) \right)^{\frac{1}{n}}$$

where  $h^-$  is the negative part of  $h$ : ( $h = h^+ - h^-$ ) and

$$S^+ = \{x \in O_M; \quad D^2(u_M - w_p) > 0 \quad \}.$$

On  $S^+$  by the concavity of  $\det^{\frac{1}{n}}$  on positive definite matrices we have

$$\det^{\frac{1}{n}}\left(\frac{D^2 u_M}{2}\right) \geq \frac{1}{2} \det^{\frac{1}{n}}(D^2(u_M - w_p)) + \frac{1}{2} \det^{\frac{1}{n}}(D^2 w_p).$$

Thus

$$\max_{O_M} h^- \leq C \left( \int_{O_M} |f^{\frac{1}{n}}(a_M^{-1}(Rx)) - f_p^{\frac{1}{n}}(a_M^{-1}(Rx))|^n dx \right)^{\frac{1}{n}}.$$

It is easy to see from (1.4) that the right hand side is  $O(1/R)$ . Next we cite the homogenization theorem of Caffarelli-Li (Theorem 3 of [9]): Let  $w$  satisfy

$$\det(D^2 w) = 1, \quad \text{in } O_M$$

with  $w = M/R^2$  on  $\partial O_M$ , then

$$|w - w_p| \leq CR^{-\delta}$$

for some  $\delta(d_0, d_1) > 0$ . Thus

$$(2.4) \quad |u_M(x) - w(x)| \leq CR^{-\delta}$$

for some  $\delta > 0$ . Set

$$E_M := \{x; \quad (x - \bar{x})' D^2 w(\bar{x})(x - \bar{x}) \leq 1\}$$

where  $\bar{x}$  is the unique minimum of  $w$  (note that Caffarelli [5] proved that the minimum point of  $w$  is unique) that satisfies  $\text{dist}(\bar{x}, \partial O_M) > C(d_1, n)$ . By the same argument in [6, 8] there exist  $\bar{k}$  and  $C$  depending only on  $n$  and  $d_1$  such that for  $\epsilon < \delta$ ,  $M = 2^{(1+\epsilon)k}$ ,  $2^{k-1} \leq M' \leq 2^k$ ,  $R \sim M^{\frac{1}{2}}$ ,

$$\left(\frac{2M'}{R^2} - C2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_M \subset \frac{1}{R} a_M(\Omega_{M'}) \subset \left(\frac{2M'}{R^2} + C2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_M, \quad \forall k \geq \bar{k},$$

which can be translated as

$$\sqrt{2M'}\left(1 - \frac{C}{2^{\epsilon k/2}}\right)E_M \subset a_M(\Omega_{M'}) \subset \sqrt{2M'}\left(1 + \frac{C}{2^{\epsilon k/2}}\right)E_M.$$

Let  $Q$  be a positive definite matrix satisfying  $Q^2 = D^2 w(\bar{x})$ ,  $O$  be an orthogonal matrix that makes  $T_k = OQa_M$  upper triangular, then  $\det(T_k) = 1$  and by Proposition 3.4 of [8]

$$\|T_k - T\| \leq C2^{-\frac{\epsilon k}{2}}$$

for some matrix  $T$ . By setting

$$v = u \cdot T$$

we have

$$\det(D^2 v(x)) = f(Tx)$$

and

$$\sqrt{2M'}\left(1 - \frac{C}{2^{\epsilon k/2}}\right)B_1 \subset \{x; \quad v(x) < M'\} \subset \sqrt{2M'}\left(1 + \frac{C}{2^{\epsilon k/2}}\right)B_1$$

for all  $M' \geq 2^{\bar{k}}$ . Here  $B_1$  stands for the unit ball (throughout the article we use  $B(p, r)$  to denote the ball centered at  $p$  with radius  $r$ . If  $p$  is the origin we may use  $B_r$ ). Consequently

$$|v(x) - \frac{1}{2}|x|^2| \leq C|x|^{2-\epsilon}.$$

The equation for  $v$  is

$$\det(D^2 v) = f_v(x)$$

where  $f_v(x) = f(T(x))$ , correspondingly we let  $f_{v,p}(x) = f_p(T(x))$ .

### Step two: Uniform Ellipticity

The purpose of this step is to show: There exist  $c_1$  and  $c_2$  depending only on  $d_0, d_1, \beta, a_1, \dots, a_n$  and  $n$  such that

$$(2.5) \quad c_1 I \leq D^2 v \leq c_2 I.$$

First we choose  $M > 100$  so that for  $R = M^{\frac{1}{2}}$  and

$$v_R(y) = \frac{1}{R^2} v(Ry),$$

$$\Omega_{1,v_R} := \{y; \quad v_R(y) \leq 1\}$$

is very close to  $B_{\sqrt{2}}$  in the sense that  $\partial\Omega_{1,v_R} \subset B_{\sqrt{2}+\epsilon} - B_{\sqrt{2}-\epsilon}$  for some  $\epsilon > 0$  small. Applying the standard interior estimate for Monge-Ampère equations ([6, 27]) we have

$$\|D^2v_R\|_{L^\infty(\Omega_{1,v_R})} = \|D^2v\|_{L^\infty(B_2)} \leq C.$$

In general for  $|x| > 100$  we shall prove (2.5) for  $D^2v(x)$ . To this end we consider the following second order incremental for  $v$ :

$$\Delta_e^2 v(x) := \frac{v(x+e) + v(x-e) - 2v(x)}{\|e\|^2}$$

where  $e \in \mathbb{R}^n$  and  $\|e\|$  is its Euclidean norm. Later we shall always choose  $e \in E$  which is defined as

$$(2.6) \quad E := \{a_1 v_1 + \dots + a_n v_n; \quad a_1, \dots, a_n \in \mathbb{Z}, \\ f_{v,p}(x + v_i) = f_{v,p}(x), \quad \forall x \in \mathbb{R}^n, \quad i = 1, \dots, n.\}$$

Let  $R_1 = |x|$  (recall  $|x| > 10$ ),  $e_x = \frac{x}{R_1}$  and

$$v_{R_1}(y) = \frac{1}{R_1^2} v(R_1 y), \quad y \in B(e_x, \frac{2}{3}).$$

Then by the closeness result of step one, the sections:  $S_{v_{R_1}}(e_x, \frac{1}{4})$  and  $S_{v_{R_1}}(e_x, \frac{1}{8})$  around  $e_x$  are very close to the corresponding sections of the parabola  $\frac{1}{2}|y|^2$ . Here we recall that for a convex,  $C^1$  function  $v$ ,

$$S_v(x, h) := \{y; \quad v(y) \leq v(x) + \nabla v(x) \cdot (y - x) + h, \quad \}.$$

In other words,  $S_{v_{R_1}}(e_x, \frac{1}{4})$  looks very similar to  $B(e_x, 1/\sqrt{2})$  and  $S_{v_{R_1}}(e_x, \frac{1}{8})$  is very close to  $B(e_x, 1/2)$ . Moreover the equation for  $v_{R_1}$  is

$$(2.7) \quad \det(D^2v_{R_1}(y)) = f_v(R_1 y)$$

where  $f_v(R_1 y) = f(T(R_1 y))$ . Let  $e_{R_1} = e/R_1$ , then direct computation shows

$$\begin{aligned} \Delta_e^2 v(x) &= \frac{v(x+e) + v(x-e) - 2v(x)}{\|e\|^2}, \quad x = R_1 y \\ &= \frac{(v(R_1 y + e) + v(R_1 y - e) - 2v(R_1 y))}{\|e\|^2} \\ &= \frac{R_1^2(v_{R_1}(y + e_{R_1}) + v_{R_1}(y - e_{R_1}) - 2v_{R_1}(y))}{\|e\|} = \Delta_{e_{R_1}}^2 v_{R_1}(y). \end{aligned}$$

Let

$$w(y) = \frac{v_{R_1}(y + e_{R_1}) + v_{R_1}(y - e_{R_1})}{2}$$

and  $F = \det^{\frac{1}{n}}$ . Then the concavity of  $F$  on positive definite matrices gives

$$\begin{aligned} \det^{\frac{1}{n}}(D^2w)(y) &\geq \frac{1}{2} \det^{\frac{1}{n}}(D^2v_{R_1}(y + e_{R_1})) + \frac{1}{2} \det^{\frac{1}{n}}(D^2v_{R_1}(y - e_{R_1})) \\ (2.8) \quad &= \frac{1}{2} f_v^{\frac{1}{n}}(R_1 y + e) + \frac{1}{2} f_v^{\frac{1}{n}}(R_1 y - e). \end{aligned}$$

On the other hand

$$(2.9) \quad \begin{aligned} F(D^2 w) &\leq F(D^2 v_{R_1}) + F_{ij}(D^2 v_{R_1}) \partial_{ij}(w - v_{R_1}) \\ &= f_v^{\frac{1}{n}}(R_1 y) + F_{ij}(D^2 v_{R_1}) \partial_{ij}(w - v_{R_1}) \end{aligned}$$

where

$$F_{ij}(D^2 v_{R_1}) = \frac{\partial F(D^2 v_{R_1})}{\partial_{ij} v_{R_1}} = \frac{1}{n} (\det(D^2 v_{R_1}))^{\frac{1}{n}-1} \text{cof}_{ij}(D^2 v_{R_1}).$$

Thus the combination of (2.8) and (2.9) gives

$$a_{ij} \partial_{ij} (\Delta_{e_{R_1}}^2 v_{R_1}) \geq E_1$$

where

$$\begin{aligned} a_{ij} &= \text{cof}_{ij}(D^2 v_{R_1}), \\ E_1 &:= n \det(D^2 v_{R_1})^{(n-1)/n} \\ &\quad \cdot \left( \frac{R_1^2}{2\|e\|^2} (f_v^{\frac{1}{n}}(R_1 y + e) + f_v^{\frac{1}{n}}(R_1 y - e) - 2f_v^{\frac{1}{n}}(R_1 y)) \right) \end{aligned}$$

For applications later we state the following fact: For any smooth  $u$

$$(2.10) \quad \partial_i (\text{cof}(D^2 u)_{ij}) = 0.$$

Let  $f_{v,p}(y) = f_p(Ty)$ , then by choosing  $e \in E$  (see (2.6))

$$(2.11) \quad f_{v,p}^{\frac{1}{n}}(R_1 y + e) + f_{v,p}^{\frac{1}{n}}(R_1 y - e) - 2f_{v,p}^{\frac{1}{n}}(R_1 y) = 0, \quad \text{for all } y \in \mathbb{R}^n.$$

By (1.4) we see that, for  $y \in B(e_x, \frac{1}{4})$  (which implies  $2 > |y| > 1 - 1/\sqrt{2}$ ) and  $e \in E$  with  $\|e\| \leq \frac{1}{10} R_1$

$$\begin{aligned} E_1(y) &= n (\det(D^2 v_{R_1}))^{(n-1)/n} \frac{R_1^2}{2\|e\|^2} \{ (f_v^{\frac{1}{n}}(R_1 y + e) - f_{v,p}^{\frac{1}{n}}(R_1 y + e)) \\ &\quad + (f_v^{\frac{1}{n}}(R_1 y - e) - f_{v,p}^{\frac{1}{n}}(R_1 y - e)) - 2(f_v^{\frac{1}{n}}(R_1 y) - f_{v,p}^{\frac{1}{n}}(R_1 y)) \} \\ &= O(R_1^{-\beta}). \end{aligned}$$

Now we construct a function  $h_0$  that solves

$$\begin{cases} a_{ij} \partial_{ij} h_0 = -E_1, & \text{in } B(e_x, \frac{1}{4}) \\ h_0 = 0 & \text{on } \partial B(e_x, \frac{1}{4}) \end{cases}$$

Then we use the following classical estimate of Aleksandrov (see Page 220-222 of [22] for a proof):

*Theorem A:* Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and let  $v$  be a solution in  $\Omega$  of the equation

$$a_{ij}^* \partial_{ij} v = g$$

such that  $v = 0$  on  $\partial\Omega$  and the coefficient matrix  $(a_{ij}^*)_{n \times n}$  satisfies

$$c_1 \leq \det(a_{ij}^*) \leq c_2, \quad \text{and} \quad (a_{ij}^*) > 0,$$

then

$$|v(x)| \leq C(n, c_1, c_2) \text{diam}(\Omega) \|g\|_{L^n(\Omega)}, \quad \forall x \in \Omega$$



Applying Theorem A to  $h_0$  we have

$$(2.12) \quad |h_0(y)| \leq C(n, d_0) R_1^{-\beta}, \quad \text{for } y \in B(e_x, \frac{1}{4}).$$

**Remark 2.1.** *The estimate of Theorem A does not depend on constants of uniform ellipticity.*

Thus  $\Delta_{e_{R_1}}^2 v_{R_1} + h_0$  is a super solution:

$$a_{ij} \partial_{ij} (\Delta_{e_{R_1}}^2 v_{R_1} + h_0) \geq 0, \quad \text{in } B(e_x, \frac{1}{4}).$$

In order to obtain a pointwise estimate for  $\Delta_{e_{R_1}}^2 v_{R_1}$  we recall the following Harnack inequality of Caffarelli-Gutierrez in [10]:

*Theorem B (Caffarelli-Gutierrez):* Let  $O$  be a convex set in  $\mathbb{R}^n$  and  $B_1 \subset O \subset B_n$  ( $n \geq 2$ ). Suppose  $\phi \in C^2(O)$  satisfies, for  $0 < \lambda < \Lambda < \infty$ ,

$$\lambda < \det(D^2 \phi) \leq \Lambda, \quad \text{in } O, \quad \text{and } \phi = 0 \quad \text{on } \partial O.$$

Then for any  $r > s > 0$  and any  $w \in C^2(O)$  satisfying

$$a_{ij} \partial_{ij} w \geq 0, \quad w \geq 0,$$

where  $a_{ij} = \det(D^2 \phi) \phi^{ij}$  and  $(\phi^{ij})_{n \times n} = (D^2 \phi)^{-1}$ , we have

$$\max_{x \in O, \text{dist}(x, \partial O) > r} w \leq C \int_{x \in O, \text{dist}(x, \partial O) > s} w,$$

for some  $C > 0$  depending only on  $\lambda, \Lambda, r, s$ .

Using Theorem B for  $\Delta_{e_{R_1}}^2 v_{R_1} + h_0$  we have

$$\max_{y \in S(v_{R_1}, e_x, \frac{1}{16})} (\Delta_{e_{R_1}}^2 v_{R_1} + h_0) \leq C(n, d_0) \int_{S(v_{R_1}, e_x, \frac{1}{8})} (\Delta_{e_{R_1}}^2 v_{R_1} + h_0)$$

Note that the distance between the two sections above is comparable to 1. It is also important to point out that the estimates in Theorem B does not depend on the regularity of  $v_{R_1}$ . Before we proceed we cite the following Calculus lemma that can be found in [9]:

*Lemma B:* Let  $g \in C^2(-1, 1)$  be a strictly convex function, and let  $0 < |h| \leq \epsilon$ . Then

$$\Delta_h^2 g(x) > 0, \quad \forall |x| \leq 1 - 2\epsilon, \quad \text{and} \quad \int_{-1+2\epsilon}^{1-2\epsilon} \Delta_h^2 g \leq \frac{C}{\epsilon} \text{osc}_{(-1,1)} g,$$

where  $C$  is a universal constant and  $\text{osc}$  stands for oscillation.

Let  $L$  be a line parallel to  $e$ , then by Lemma B,

$$\int_{S(v_{R_1}, e_x, \frac{1}{8}) \cap L} \Delta_{e_{R_1}}^2 v_{R_1} \leq C.$$

Consequently, letting  $L$  go through all directions, we have

$$\int_{S(v_{R_1}, e_x, \frac{1}{8})} \Delta_{e_{R_1}}^2 v_{R_1} \leq C.$$

Since  $h_0 = O(R_1^{-\beta})$  we have proved

$$(2.13) \quad 0 \leq \Delta_e^2 v(x) = \Delta_{e_{R_1}}^2 v_{R_1}(y) \leq C, \quad \text{for } y \in B(e_x, \frac{1}{16}), \quad x = R_1 y.$$

Note that  $\Delta_e^2 v \geq 0$  because  $v$  is convex. Then the same argument as in [9] can be employed to prove that the level surfaces of  $v$  are like balls.

For the convenience of the readers we describe the outline of this argument. Given  $x \in \mathbb{R}^n$  with  $|x| > 100$  we set

$$\gamma := \sup_{e \in E, \|e\| \leq \frac{1}{10}\|x\|} \sup_{y \in \mathbb{R}^n} \Delta_e^2 v(y)$$

and

$$\bar{v}(z) = v(z + x) - v(x) - \nabla v(x)z.$$

Clearly  $\bar{v}(0) = 0 = \min_{\mathbb{R}^n} \bar{v}$ . By (2.13) it is easy to see

$$\sup_{B_r} \bar{v} \leq C(n)\gamma r^2, \quad \forall r \in (1, \frac{1}{10}\|x\|).$$

On the other hand for any  $\bar{z} \in \partial B_r$  we show that for  $r$  large (but still only depending on  $n$  and  $d_0$ ),  $\bar{v}(\bar{z}) \geq 1$ . Indeed, by (2.13),

$$(2.14) \quad \bar{v}(\frac{\bar{z}}{2} + e) + \bar{v}(\frac{\bar{z}}{2} - e) - 2\bar{v}(\frac{\bar{z}}{2}) \leq \gamma\|e\|^2$$

for all  $e \in E$  and  $\|e\| \leq \frac{1}{10}\|x\|$ . Hence for  $z \in \frac{\bar{z}}{2} + (-2, 2)^n$ , we find  $e \in E$  with  $\|e\| \leq C(n)$  so that  $z$  is on the same line with  $\frac{\bar{z}}{2} + e$  and  $\frac{\bar{z}}{2} - e$  and is between them. Thus (2.14) implies

$$\bar{v}(z) \leq \bar{v}(\frac{\bar{z}}{2} + e) + \bar{v}(\frac{\bar{z}}{2} - e) \leq 2\bar{v}(\frac{\bar{z}}{2}) + C(n)\gamma.$$

Further more, by  $\bar{v}(0) = 0$  and the convexity of  $\bar{v}$  we have

$$2\bar{v}(\frac{\bar{z}}{2}) \leq \bar{v}(\bar{z}).$$

Therefore the following holds:

$$\bar{v}(z) \leq \bar{v}(\bar{z}) + C(n)\gamma, \quad z \in \frac{\bar{z}}{2} + (-2, 2)^n.$$

Consider

$$w(z) = \frac{\bar{v}(\frac{\bar{z}}{2} + z)}{\bar{v}(\bar{z}) + C(n)\gamma}.$$

Clearly  $w$  satisfies  $\det(D^2 w) \geq d_0^{-1}/(\bar{v}(\bar{z}) + C(n)\gamma)^n$  and

$$w(z) \leq 1, \quad \text{for } z \in \bar{z}/2 + (-2, 2)^n.$$

Applying Lemma A we have

$$\max\{1, \bar{v}(\bar{z})\} \geq \left( \frac{d_0^{-\frac{1}{n}} C(n)}{\bar{v}(\bar{z}) + C(n)\gamma} \right) r^{\frac{2}{n}}.$$

Here we used the fact that  $\bar{v}(0) = 0$  and  $\bar{v}(\bar{z})$  is the maximum value of  $\bar{v}$  on the line segment connecting 0 and  $\bar{z}$ . If  $\bar{v}(\bar{z}) \geq \gamma$ , no need to do anything (here we assume  $\gamma > 1$  without loss of generality), otherwise we obviously have

$$\max\{1, \bar{v}(\bar{z})\} \geq \frac{d_0^{-\frac{1}{n}}}{C(n)\gamma} r^{\frac{2}{n}}.$$

Choose  $r$  large but only depending on  $n, \gamma$  and  $d_0$  we can still make the right hand side of the above greater than 1. Therefore we have proved  $\bar{v}(z) \geq 1$ . By standard argument the sections of  $\bar{v}$ :  $S(\bar{v}, 0, h)$  are similar to balls if  $h \sim 1$  and  $h < 1$ . Then interior estimate of Caffarelli, Jian-Wang [27] can be employed on  $S(\bar{v}, 0, h)$  to obtain the  $C^{2,\alpha}$  norm of  $\bar{v}$ . In particular  $|D^2 v(x)| = |D^2 \bar{v}(x)| \leq C$ . Once the upper bound is obtained we also have the lower bounded from the Monge-Ampère equation. (2.5) is established.

**Remark 2.2.** *In order to prove (2.5) the assumptions on the derivatives of  $f^{\frac{1}{n}} - f_p^{\frac{1}{n}}$  are not essential. In fact, as long as*

$$|(f^{\frac{1}{n}} - f_p^{\frac{1}{n}})(x)| \leq C(1 + |x|)^{-\beta}, \quad \text{for } x \in \mathbb{R}^n,$$

(2.5) still holds.

### Step three. Pointwise estimate of $\Delta_e^2 v$

Let  $e \in E$ , recall the equation for  $\Delta_e^2 v$  is

$$(2.15) \quad \tilde{a}_{ij}(x) \partial_{ij} \Delta_e^2 v \geq E_2.$$

where

$$\tilde{a}_{ij}(x) = \text{cof}_{ij}(D^2 v(x))$$

is uniformly elliptic and divergence free,

$$E_2 = n(\det(D^2 v))^{\frac{n-1}{n}} \frac{f_v^{\frac{1}{n}}(x+e) + f_v^{\frac{1}{n}}(x-e) - 2f_v^{\frac{1}{n}}(x)}{\|e\|^2}.$$

By the assumption on  $f$ ,

$$\begin{aligned} |E_2(x)| &= n(\det(D^2 v))^{\frac{n-1}{n}} \\ &\quad \left( \frac{(f_v^{\frac{1}{n}}(x+e) - f_{p,v}^{\frac{1}{n}}(x+e)) + (f_v^{\frac{1}{n}}(x-e) - f_{v,p}^{\frac{1}{n}}(x-e))}{\|e\|^2} \right. \\ &\quad \left. - 2 \frac{(f_v^{\frac{1}{n}}(x) - f_{v,p}^{\frac{1}{n}}(x))}{\|e\|^2} \right). \end{aligned}$$

Here we have the following important observation:

$$(2.16) \quad |E_2(x)| \leq C(1 + |x|)^{-2-\beta}, \quad \forall e \in E$$

where  $C$  is independent of  $e \in E$  and  $x$ . Indeed, let  $g(x) = f_v^{\frac{1}{n}}(x) - f_{v,p}^{\frac{1}{n}}(x)$ , then

$$|E_2(x)| = (g(x+p) + g(x-p) - 2g(x))/\|e\|^2.$$

Without loss of generality we assume that  $e = (\|e\|, 0, \dots, 0)$ . Then

$$\begin{aligned} E_2(x) &= \frac{(g(x+e) - g(x)) - (g(x) - g(x-e))}{\|e\|^2} \\ &= \frac{\int_0^{\|e\|} \partial_1 g(x_1 + t, x') dt - \int_0^{\|e\|} \partial_1 g(x_1 - \|e\| + t, x') dt}{\|e\|^2} \\ &= \frac{\int_0^{\|e\|} \int_0^{\|e\|} \partial_{11} g(x_1 - \|e\| + t + s, x') dt ds}{\|e\|^2} \\ &= \int_0^1 \int_0^1 \partial_{11} g(x_1 - \|e\| + \|e\|t + \|e\|s, x') dt ds. \end{aligned}$$

Using

$$|D^2 g(x)| \leq C(1 + |x|)^{-\beta-2},$$

which comes from (1.4) we have

$$|E_2(x)| \leq C \int_0^1 \int_0^1 \left( (x_1 + \|e\|(-1 + t + s))^2 + |x'|^2 \right)^{-\frac{\beta+2}{2}} dt ds.$$

If  $|x'| > \frac{1}{8}|x|$ , (2.16) holds obviously. So we only consider when  $|x_1| > \frac{1}{2}|x|$ .

$$\begin{aligned} & \int_0^1 \int_0^1 \left( (x_1 + \|e\|(-1 + t + s))^2 + |x'|^2 \right)^{-\frac{\beta+2}{2}} dt ds \\ & \leq \int_0^1 \int_0^1 \left| x_1 + \|e\|(-1 + t + s) \right|^{-\beta-2} dt ds \\ & = \|e\|^{-\beta-2} \int_0^1 \int_0^1 |L - 1 + t + s|^{-\beta-2} dt ds, \quad L = x_1/\|e\|. \end{aligned}$$

If  $|L| > 8$ ,  $|L - 1 + t + s| > |L|/2$ , which gives

$$\begin{aligned} & \int_0^1 \int_0^1 \left( (x_1 + \|e\|(-1 + t + s))^2 + |x'|^2 \right)^{-\frac{\beta+2}{2}} dt ds \\ & \leq C\|e\|^{-\beta-2}|L|^{-\beta-2} \leq C\|x_1\|^{-\beta} \leq C|x|^{-\beta-2}. \end{aligned}$$

where all the constants are absolute constants. Clearly (2.16) holds in this case.

If  $|L| < 8$ , easy to see that

$$\int_0^1 \int_0^1 |L - 1 + t + s|^{-\beta-2} dt ds \leq C(\beta).$$

Then

$$\begin{aligned} & \int_0^1 \int_0^1 \left( (x_1 + \|e\|(-1+t+s))^2 + |x'|^2 \right)^{-\frac{\beta+2}{2}} dt ds \\ & \leq C(\beta) \|e\|^{-\beta-2} \leq C(\beta) \|x\|^{-\beta-2}, \end{aligned}$$

since  $|L| < 8$  implies  $\|e\| > |x_1|/8 > |x|/16$ . (2.16) is established.

By standard elliptic estimates  $f \in C^{1,\alpha}$  yields  $\tilde{a}_{ij} \in C^{1,\alpha}$ . Moreover  $\tilde{a}_{ij}$  is divergence free. The Green's function  $G(x, y)$  corresponding to  $-\partial_{x_i}(\tilde{a}_{ij}\partial_{x_j})$  satisfies

$$-\partial_{x_i}(\tilde{a}_{ij}\partial_{x_j}G(x, y)) = \delta_y$$

and

$$(2.17) \quad 0 \leq G(x, y) \leq C|y - x|^{2-n}, \quad x, y \in \mathbb{R}^n$$

by a result of Littman-Stampacchia-Weinberger in [33].

Let  $E_2^- \geq 0$  be the negative part of  $E_2$ :  $E_2 = E_2^+ - E_2^-$  and

$$h(x) = - \int_{\mathbb{R}^n} G(x, y) E_2^-(y) dy.$$

Then  $h(x)$  satisfies

$$(2.18) \quad \partial_{x_i}(\tilde{a}_{ij}\partial_{x_j})h = E_2^-, \quad \text{in } \mathbb{R}^n.$$

Here we claim that  $h$  satisfies

$$(2.19) \quad 0 \leq -h(x) \leq C(1 + |x|)^{-\beta_1}, \quad \beta_1 = \min\{n-2, \beta\}$$

The estimate of (2.19) is rather standard, for  $x \in \mathbb{R}^n$ , we divide  $\mathbb{R}^n$  into three regions:

$$\begin{aligned} \Omega_1 &= \{y; |y - x| < |x|/2\} \\ \Omega_2 &= \{y; |y| < |x|/2\}, \\ \Omega_3 &= \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2). \end{aligned}$$

Then one observes  $\int_{\Omega_i} G(x, y) E_2^-(y) dy$  for each  $\Omega_i$  can be estimated easily. (2.19) is established. From the definition of  $h$  we see that

$$(2.20) \quad \tilde{a}_{ij}\partial_{ij}(\Delta_e^2 v + h) \geq 0, \quad \text{in } \mathbb{R}^n.$$

The main result of this step is:

$$(2.21) \quad \text{Given } e \in E; \quad \Delta_e^2 v(x) \leq 1 - h(x), \quad \text{for } x \in \mathbb{R}^n.$$

Let  $v_\lambda(x) = v(\lambda x)/\lambda^2$  and  $P(x) = \frac{1}{2}|x|^2$ . First we claim that

$$(2.22) \quad D^j(v_\lambda - P(x)) \rightarrow 0, \quad j = 0, 1, \quad \forall x \in K \subset \subset \mathbb{R}^n, \text{ as } \lambda \rightarrow \infty,$$

where  $K$  is any fixed compact subset of  $\mathbb{R}^n$ .

To see (2.22), by step one

$$(2.23) \quad |v_\lambda(x) - P(x)| \leq C\lambda^{-\epsilon}, \quad \forall x \in K \subset \subset \mathbb{R}^n.$$

and  $|D^2v_\lambda(x)| \leq C$  for  $x \in K$  we obtain by Ascoli's theorem that  $\partial_l v_\lambda(x)$  ( $l = 1, \dots, n$ ) tends to a continuous function. By (2.23) this function has to be  $x_l$ . Thus (2.22) holds.

**Remark 2.3.** *We don't have the estimate of  $\|Dv_\lambda - x\|_{L^\infty(K)}$  but we don't need it.*

The proof of (2.21) is as follows.

Let  $\alpha = \sup_{\mathbb{R}^n} (\Delta_e^2 v + h)$  for  $e \in E$ . By step two  $\alpha < \infty$ . Let  $\hat{e} = e/\lambda$ , then by (2.22) and Lebesgue's dominated convergence theorem

$$(2.24) \quad \lim_{\lambda \rightarrow \infty} \int_{B_1} \Delta_{\hat{e}}^2 v_\lambda = \int_{B_1} 1 dx = |B_1|.$$

Indeed, the integral over  $B_1$  can be considered as the collection of integration on segments all in the direction of  $\hat{e}$ . Since  $Dv_\lambda \rightarrow DP$  in  $C^0$  and  $DP$  is smooth, the following Lemma C in [9] implies (2.24):

*Lemma C (Caffarelli-Li) Let  $g_i$  converge to  $g$  in  $C^1[-1, 1]$ ,  $g \in C^2(-1, 1)$  and  $|h_i| \rightarrow 0$  as  $i \rightarrow \infty$ . Then for all  $-1 < a < b < 1$ ,*

$$\lim_{i \rightarrow \infty} \int_a^b \Delta_{h_i}^2 g_i = g'(b) - g'(a) = \int_a^b g''.$$

Let

$$h_\lambda(x) = h(\lambda x)/\lambda^2.$$

It follows immediately from (2.19) that

$$\lim_{\lambda \rightarrow \infty} \int_{B_1} (\Delta_{\hat{e}}^2 v_\lambda + h_\lambda) = |B_1|,$$

which obviously implies  $\alpha \geq 1$  because otherwise it is easy to see that the left hand side of the above is less than  $|B_1|$ . Our goal is to show that  $\alpha = 1$ . If this is not the case we have  $\alpha > 1$ . Then

$$\limsup_{\lambda} \left( \frac{\alpha + 1}{2} |\{\Delta_{\hat{e}}^2 v_\lambda + h_\lambda \geq \frac{\alpha + 1}{2}\} \cap B_1| \right) \leq \lim_{\lambda \rightarrow \infty} \int_{B_1} \Delta_{\hat{e}}^2 v_\lambda + h_\lambda = |B_1|.$$

Thus

$$\frac{|\{\Delta_{\hat{e}}^2 v_\lambda + h_\lambda \geq \frac{\alpha + 1}{2}\} \cap B_1|}{|B_1|} \leq 1 - \mu$$

for  $\mu = \frac{1}{2}(1 - \frac{2}{\alpha + 1}) > 0$ . Here we emphasize that it is important to have  $\mu > 0$ . Equivalently for large  $\lambda$

$$\frac{|\{\Delta_{\hat{e}}^2 v_\lambda + h_\lambda \leq \frac{\alpha + 1}{2}\} \cap B_1|}{|B_1|} \geq \mu.$$

Then we cite the following well known result which can be found in [7] (Lemma 6.5):

*Theorem C: For  $v$  satisfying*

$$a_{ij}^* \partial_{ij} v \geq 0, \quad \lambda I \leq (a_{ij}^*(x)) \leq \Lambda I$$

and

$$v \leq 1 \quad \text{in } B_1 \quad \text{and} \quad |\{v \leq 1 - \epsilon\} \cap B_1| \geq \mu|B_1|,$$

then  $v \leq 1 - c(n, \lambda, \Lambda, \epsilon, \mu)$  over  $B_{1/2}$

By (2.20) we see that

$$\tilde{a}_{ij}(\lambda) \partial_{ij}(\Delta_{\epsilon}^2 v_{\lambda} + h_{\lambda}) \geq 0, \quad \text{in } B_1.$$

Applying Theorem C to  $\Delta_{\epsilon}^2 v_{\lambda} + h_{\lambda}$  we have

$$(2.25) \quad \Delta_{\epsilon}^2 v_{\lambda} + h_{\lambda} \leq \alpha - C, \quad \text{in } B_{1/2}.$$

Since  $\Delta_{\epsilon}^2 v_{\lambda} + h_{\lambda}$  is sub-harmonic,

$$\alpha = \sup_{\mathbb{R}^n} \Delta_{\epsilon}^2 v + h = \lim_{\lambda \rightarrow \infty} \sup_{B_{1/2}} (\Delta_{\epsilon}^2 v_{\lambda} + h_{\lambda}) < \alpha.$$

Then we get a contradiction, (2.21) is established.

**Step four: The proof of the Liouville theorem for  $n \geq 4$ .**

By a result of Li [32] there exists  $\xi \in C^{2,\alpha}(\mathbb{R}^n)$  such that

$$\begin{cases} \det(I + D^2 \xi) = f_{v,p}, & \text{in } \mathbb{R}^n, \\ I + D^2 \xi > 0, \end{cases}$$

where  $\xi$  is a periodic function with the same period as that of  $f_{v,p}$ . Let  $P(x) = \frac{1}{2}|x|^2$  and

$$w(x) = v(x) - P(x) - \xi(x).$$

Using  $\xi(x + e) = \xi(x)$  for all  $x \in \mathbb{R}^n$  and  $e \in E$ ,  $\Delta_{\epsilon}^2 P = 1$  and (2.21) we have

$$(2.26) \quad \Delta_{\epsilon}^2 w + h \leq 0, \quad \forall x \in \mathbb{R}^n, \quad \forall e \in E.$$

From the equations for  $v$  and  $P + \xi$  we have,

$$\det(D^2 v) - \det(D^2(P + \xi)) = f_v - f_{v,p}, \quad \text{in } \mathbb{R}^n,$$

which gives

$$\hat{a}_{ij} \partial_{ij} w = f_v - f_{v,p} \quad \text{in } \mathbb{R}^n$$

where

$$\hat{a}_{ij}(x) = \int_0^1 \text{cof}_{ij}(tD^2 v + (1-t)D^2(P + \xi)) dt$$

is uniformly elliptic and divergence free. Using the Green's function corresponding to  $-\partial_i(\hat{a}_{ij}(x)\partial_j)$  we find  $h_3$  to solve

$$\hat{a}_{ij} \partial_{ij} h_3 = f_{v,p} - f_v, \quad \text{in } \mathbb{R}^n$$

and

$$h_3(x) = O((1 + |x|)^{-\beta_1}), \quad x \in \mathbb{R}^n.$$

Therefore we have

$$(2.27) \quad \hat{a}_{ij} \partial_{ij}(w + h_3) = 0, \quad \text{in } \mathbb{R}^n, \quad n \geq 3.$$

Next we let

$$M_k = \sup_{B(0,k)} (w + h_3).$$

Then we claim that

$$(2.28) \quad M_k \leq 2M_{k/2} + C.$$

for  $C > 0$  independent of  $k$ .

In order to prove (2.28) we consider the equation for  $\Delta_e^2 w$  where  $e \in E$ . In view of (2.18) we have

$$\tilde{a}_{ij} \partial_{ij} (\Delta_e^2 w) \geq E_2, \quad \text{in } \mathbb{R}^n.$$

where  $\tilde{a}_{ij}(x) = \text{cof}_{ij}(D^2 v)$  is divergence free and  $|E_2(x)| \leq C(1 + |x|)^{-2-\beta}$ .

Just like the estimate of  $h$  we can find  $h_2$  that satisfies

$$\tilde{a}_{ij} \partial_{ij} (\Delta_e^2 w + h_2) \geq 0 \quad \text{in } \mathbb{R}^n$$

and

$$|h_2(x)| \leq C(1 + |x|)^{-\beta_1}, \quad \text{in } \mathbb{R}^n.$$

Let  $x_0$  be where  $\Delta_e^2 w + h_2$  attains its maximum on  $B_k$ , clearly  $x_0 \in \partial B_k$ . Let  $e = \frac{x_0}{2} + a$  be chosen so that  $e \in E$  and  $|a|$  is small:  $|a| \leq C$ . Then for  $k$  large

$$\Delta_e^2 w(x - e) \leq Ck^{-\beta_1}$$

which is

$$(2.29) \quad w(x_0) \leq 2w(x_0 - e) - w(x_0 - 2e) + O(k^{2-\beta_1}).$$

Since  $|x_0 - 2e|$  is in the neighborhood of 0, we have  $|w(x_0 - 2e)| \leq C$  and

$$w(x_0) \leq 2 \max_{B(0,k/2)} w + C.$$

Hence (2.28) is verified.

Set

$$g_k(y) = (w(ky) + h_3(ky))/M_k, \quad |y| \leq 1.$$

It follows from (2.28),  $w_1(0) = 0$  and  $h_3 = O(1)$  that

$$\max_{B_1} g_k \rightarrow 1, \quad \max_{B_{1/2}} g_k \geq \frac{1}{4}, \quad g_k(0) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Then we observe that the Harnack inequality holds for  $1 - g_k$  in  $B_1$  because  $1 - g_k \geq 0$  in  $B_1$  and

$$\hat{a}_{ij}(k \cdot) \partial_{ij} (1 - g_k) = 0 \quad \text{in } B_1.$$

The uniform ellipticity of  $(\hat{a}_{ij})$  implies that

$$(2.30) \quad \max_K (1 - g_k) \leq C(K) \min(1 - g_k), \quad \forall K \subset\subset B_1.$$

Consequently  $g_k$  converges in  $C^\alpha$  norm to  $g$  in  $B_1$  for some  $\alpha > 0$  depending only on the upper bound and the lower bound of the eigenvalues of  $(\hat{a}_{ij})$ . Since the constant in (2.30) does not depend on  $k$ , we also have

$$\max_K (1 - g) \leq C(K) \min_K (1 - g), \quad \forall K \subset\subset B_1.$$



Next we observe that (2.29) also holds for all  $e \in E$  with  $\|e\| \leq \frac{3}{4}k$ , thus

$g$  is concave in  $B_1$

because the perturbation term disappears as  $k \rightarrow \infty$ . Let  $l$  be a linear function that touches  $g$  from above around 0 in  $B_{1/2}$ . Then  $l - g$  is a convex function that takes its minimum at the origin. Since  $l - g$  satisfies Harnack inequality,  $l - g \equiv 0$  in  $B_1$  (by the  $C^\alpha$  convergence from  $g_k$  to  $g$ , one can find  $l_k \rightarrow l$  such that  $l_k - g_k \geq 0$  in  $B_{3/4}$  and  $(l_k - g_k)(0) \rightarrow 0$ . Thus applying Harnack inequality to  $l_k - g_k$  we see that  $l - g$  satisfies Harnack inequality too.

Since  $\max_{B_1} g \geq \frac{3}{4}$  we see that  $l = a \cdot x$  for  $a \neq 0$ ,  $|a| \geq 3/4$ . In other words

$$\frac{w(ky)}{M_k} - ay = o(1), \quad |y| \leq \frac{3}{4}.$$

Let  $e \in E$  be a direction that  $w(ke) \geq \frac{1}{2}M_k k\|e\|$  for  $k$  large. Subtract a linear function from  $w$  ( $w_1 = w -$  the linear function) to make

$$w_1(0) = w_1(e) = 0.$$

Clearly

$$(2.31) \quad w_1(ke) \geq \frac{1}{4}M_k k\|e\|$$

for  $k$  large. Using

$$\Delta_e^2 w_1(x) \leq c_0 |x|^{-\beta_1}, \quad |x| \geq 1$$

we have

$$w_1(2e) \leq 2w_1(e) - w_1(0) + c_0 \|e\|^2 \|e\|^{-\beta_1} = c_0 \|e\|^{2-\beta_1}.$$

Similarly

$$\begin{aligned} w_1(4e) &\leq 2w_1(2e) - w_1(0) + c_0 \|2e\|^2 \|2e\|^{-\beta} \\ &\leq 2c_0 \|e\|^{2-\beta_1} + c_0 (2\|e\|)^{2-\beta_1}. \end{aligned}$$

In general

$$\begin{aligned} (2.32) \quad w_1(2^{N+1}e) &\leq \|e\|^{2-\beta_1} c_0 (2^N + 2^{N-1+(2-\beta_1)} + 2^{N-2+2(2-\beta_1)} + \dots + 2^{N(2-\beta_1)}) \\ &= \|e\|^{2-\beta} c_0 2^N \frac{1 - 2^{-(N+1)(\beta_1-1)}}{1 - 2^{1-\beta_1}} \end{aligned}$$

if  $\beta_1 > 1$ . Here we recall that  $\beta_1 = \min\{n-2, \beta\}$ . Thus for  $n \geq 4$  we obviously have

$$w_1(2^{N+1}e) \leq C \|2^{N+1}e\|,$$

which leads to a contradiction (2.31). Therefore we have proved that  $M_k$  is bounded if the Monge-Ampère equation is defined in  $\mathbb{R}^n$  for  $n \geq 4$ . Since  $w + h_3$  is bounded from above, it is easy to see that  $w + h_3 \equiv C$  by Harnack inequality. Theorem 1.1 is established for  $n \geq 4$ .

**Step five: Proof of Theorem 1.1 for  $n = 3$**

In  $\mathbb{R}^3$  we have  $\beta_1 = 1$  which means (2.32) becomes

$$w_1(2^{N+1}e) \leq CN(2^{N+1}\|e\|)$$

which leads to a contradiction if we assume  $M_k \geq (\log k)^2$ . Therefore the same argument has proved that

$$w(x) + h_3(x) \leq C(\log(2 + |x|))^2, \quad x \in \mathbb{R}^3.$$

In order to finish the proof we need the following

**Lemma 2.1.** *Let  $u$  solve*

$$a_{ij}^* \partial_{ij} u = 0, \quad \text{in } \mathbb{R}^n$$

where  $\lambda I \leq (a_{ij}^*(x)) \leq \Lambda I$  for all  $x \in \mathbb{R}^n$  and

$$|u(x)| \leq C(1 + |x|)^\delta, \quad x \in \mathbb{R}^n.$$

There exists  $\epsilon_0(\lambda, \Lambda) > 0$  such that if  $\delta \in (0, \epsilon_0)$   $u \equiv \text{constant}$ .

**Proof of Lemma 2.1:** For  $R > 1$  let

$$u_R(y) = u(Ry)/R^2, \quad |y| \leq 1.$$

Then  $u_R$  satisfies

$$a_{ij}^*(Ry) \partial_{ij} u_R = 0, \quad \text{in } B_1$$

and

$$|u_R(y)| \leq CR^{\delta-2}, \quad \text{in } B_1.$$

By Krylov-Safonov's [31] estimate

$$(2.33) \quad \frac{|u_R(y) - u_R(0)|}{|y|^{\epsilon_0}} \leq CR^\delta/R^2, \quad \forall y \in B_{1/2}$$

where  $\epsilon_0 > 0$  only depends on  $\lambda$  and  $\Lambda$ . Clearly (2.33) can be written as

$$|u(x) - u(0)| \leq C|x|^{\epsilon_0} R^{\delta-\epsilon_0}, \quad \forall x \in B_{R/2}.$$

Fix any  $x \in \mathbb{R}^n$ , we let  $R \rightarrow \infty$ , then  $u(x) = u(0)$  for  $\delta < \epsilon_0$ . Lemma 2.1 is established.  $\square$ .

Applying Lemma 2.1 to  $w + h_3$  we see that  $w + h_3 \equiv c$  in  $\mathbb{R}^3$ . Theorem 1.1 is also proved for the equation defined in  $\mathbb{R}^3$ .  $\square$

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